PMM U.S.S.R., Vol.51, No.1, pp.50-56, 1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

THE PROBLEM OF A CURVED LAYER OF COMPOSITE MATERIAL WITH WAVY SURFACES OF PERIODIC STRUCTURE*

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The state of stress and strain for a curved anisotropic inhomogeneous thin layer of periodic structure of variable thickness determined by wavy surfaces is investigated. The period of the structure and the wave dimensions on the surfaces are considered to be commensurate with the layer thickness in magnitude.

A general averaging scheme is used /1, 2/, as is a two-scale expansion method /3/, which enables a passage to be made from the three-dimensional to the two-dimensional equations of elasticity theory. The effective stiffness moduli of the average shell obtained are here determined from the solution of auxiliary local problems in the periodicity cell. The model constructed enables the state of stress and strain of shells of composite materials with an arbitrary kind of periodic structure reinforcement (hummocks, waffles, ribs, or corrugations) to be investigated. In particular, it is possible to consider shells from a material with a stiff skeleton of periodic structure or stiff filaments, reinforcement from a material with properties different from the properties of the host layer material, and periodically perforated shells. In the limit case of "smooth" surfaces and a homogeneous material, the well-known model of an anisotropic shell is obtained. The proposed averaging method does not require replacement of the reinforcements by a certain special kind of contact forces /4/. This enables the influence of different reinforcements to be taken into account more rigorously and also enables the stresses to be evaluated directly at the points of the periodicity cell.

1. We introduce a triorthogonal dimensionless coordinate system $\alpha_1, \alpha_2, \gamma$ such that the coordinate lines α_1 and α_2 agree with the lines of principal curvature of the middle surface (for $\gamma = 0$), while the γ axis is directed along its normal. The Lame coefficients in such a coordinate system $H_1 = A_1(1 + k_1\gamma)$, $H_2 = A_2(1 + k_2\gamma)$, $H_3 = 1$; $A_1(\alpha)$, $A_2(\alpha)$ are the coefficients of the first quadratic form, and $k_1(\alpha)$, $k_2(\alpha)$ are the principal middle surface curvatures while $\alpha = (\alpha_1, \alpha_2)$.

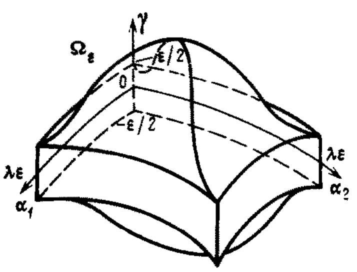

Fig.1

The elastic layer under consideration has a periodic structure with periodicity cell Ω_ε (sketch)

$$\{0 < \alpha_1 < \lambda\varepsilon,\ 0 < \alpha_2 < \lambda\varepsilon,\ \gamma^- < \gamma < \gamma^+\}$$

$$\gamma^\pm = \pm\frac{\varepsilon}{2} \pm \lambda\varepsilon F^\pm\left(\frac{\alpha_1}{\lambda\varepsilon}, \frac{\alpha_2}{\lambda\varepsilon}\right)$$

The dimensionless small parameter ε determines the layer thickness, characterizes the ratio between the periodicity cell dimensions and the layer thickness and is assumed to be a constant of the order of unity, F^+ and F^- are different functions, in the general case, which define the upper and lower layer surfaces.

The physical components of the strain tensor and the displacement vector are connected by the relationships /5/

$$e_{11} = \frac{1}{H_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{1}{H_1 A_2}\frac{\partial A_1}{\partial \alpha_2} u_2 + \frac{A_1 k_1}{H_1} u_3 (1 \leftrightarrow 2), \quad e_{33} = \frac{\partial u_3}{\partial \gamma} \tag{1.1}$$

$$2e_{12} = \frac{1}{H_2}\frac{\partial u_1}{\partial \alpha_2} - \frac{1}{H_2 A_1}\frac{\partial A_2}{\partial \alpha_1} u_2 + \frac{1}{H_1}\frac{\partial u_2}{\partial \alpha_1} - \frac{1}{H_1 A_2}\frac{\partial A_1}{\partial \alpha_2} u_1$$

$$2e_{31} = \frac{\partial u_1}{\partial \gamma} + \frac{1}{H_1}\frac{\partial u_3}{\partial \alpha_1} - \frac{A_1 k_1}{H_1} u_1 \quad (1 \leftrightarrow 2)$$

The equilibrium equations /6/ can be written in the form (P_i are the bulk force components)

*Prikl.Matem.Mekhan., 51, 1, 68-75, 1987

$$\frac{\partial (H_2\sigma_{11})}{\partial\alpha_1}+\frac{\partial (H_1\sigma_{12})}{\partial\alpha_2}+\frac{\partial (H_1H_2\sigma_{13})}{\partial\gamma}-\frac{H_1}{A_1}\frac{\partial A_2}{\partial\alpha_1}\sigma_{22}+ \tag{1.2}$$
$$\frac{H_2}{A_2}\frac{\partial A_1}{\partial\alpha_2}\sigma_{21}+H_2A_1k_1\sigma_{31}+H_1H_2P_1=0 \quad (1\leftrightarrow 2)$$
$$\frac{\partial (H_2\sigma_{31})}{\partial\alpha_1}+\frac{\partial (H_1\sigma_{32})}{\partial\alpha_2}+\frac{\partial (H_1H_2\sigma_{33})}{\partial\gamma}-H_2A_1k_1\sigma_{11}-$$
$$H_1A_2k_2\sigma_{22}+H_1H_2P_3=0$$

The stresses and strains are related by Hooke's law

$$\sigma_{ij}=a_{ijln}(y_1,y_2,z)e_{ln},\quad y_\mu=\alpha_\mu/\lambda\varepsilon,\quad z=\gamma/\varepsilon \tag{1.3}$$

Here and henceforth $i, j, l, n = 1, 2, 3$; $\mu, \nu, \beta, \delta = 1, 2$; and summation is over identical subscripts, $y = (y_1, y_2)$.

We consider the elastic layer being investigated to be fabricated from a composite material possessing the periodicity cell Ω_ε, and therefore, the elasticity coefficients $a_{ijln}(y, z)$ are singly-periodic functions in the variables y_1 and y_2.

The conditions

$$H_2\sigma_{i1}n_1^\pm+H_1\sigma_{i2}n_2^\pm+H_1H_2\sigma_{i3}n_3^\pm=H_1H_2p_i^\pm\ (\gamma=\gamma^\pm) \tag{1.4}$$
$$n_i^\pm=\left\{\mp\frac{\partial F^\pm}{\partial y_1},\mp\frac{\partial F^\pm}{\partial y_2},1\right\}\left[1+\frac{1}{H_1^2}\left(\frac{\partial F^\pm}{\partial y_1}\right)^2+\frac{1}{H_2^2}\left(\frac{\partial F^\pm}{\partial y_2}\right)^2\right]^{-1/2}$$

are satisfied on the upper and lower surfaces of the layer, i.e., for $\gamma=\gamma^\pm$, where $n_i^\pm$ are unit normals, and $p_i^\pm$ are external load components on the surfaces $\gamma=\gamma^\pm$.

2. We determine the solution of problem (1.1)-(1.4) in the form of the asymptotic expansions /1, 2/

$$u_l=u_l^{(0)}(\alpha)+\varepsilon u_l^{(1)}(\alpha,y,z)+\varepsilon^2u_l^{(2)}(\alpha,y,z)+\ldots \tag{2.1}$$

Here $u_l^{(m)}(\alpha, y, z)$ $(m = 1, 2, \ldots)$ are singly-periodic functions in y_μ.

Taking into account that the layer thickness is small compared with the middle surface radii of curvature, we introduce the notation

$$k_1=\varepsilon K_2(\alpha),\quad k_2=\varepsilon K_1(\alpha),\quad k_1+k_2=\varepsilon K_3(\alpha) \tag{2.2}$$

For the external forces we set

$$p_\nu^\pm=\pm\varepsilon^2r_\nu^\pm(\alpha,y),\ p_3^\pm=\pm\varepsilon^3q_3^\pm(\alpha,y) \tag{2.3}$$
$$P_\nu=\varepsilon f_\nu(\alpha,y,z),\quad P_3=\varepsilon^2g_3(\alpha,y,z) \tag{2.4}$$

All the functions defined in (2.3) and (2.4) are periodic in y_1 and y_2 with the periodicity cell $\Omega:\{0<y_1<1,\ 0<y_2<1,\ z^-<z<z^+\}$, $z^\pm=\pm{}^1/_2\pm\lambda F^\pm(y)$. The external tangential loads working in tension or shear in relationships (2.3) are of order ε^2 while the loads bending the layer are of order ε^3. The order of the bulk forces in (2.4) is one lower than their corresponding surface loads since it increases by one on integration over the layer thickness.

To simplify the subsequent calculations, we use the notation $\xi_1=A_1y_1$, $\xi_2=A_2y_2$ and we define the differential operators acting on the arbitrary components $t_{l\mu}$ by means of the formulas

$$\mathbf{B}_1(t_{l\mu})=\frac{1}{A_1A_2}\left[\frac{\partial (A_2t_{11})}{\partial\alpha_1}+\frac{\partial (A_1t_{21})}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}t_{12}-\frac{\partial A_2}{\partial\alpha_1}t_{22}\right] \tag{2.5}$$
$$\mathbf{K}_1(t_{l\mu})=K_2t_{31}\quad (1\leftrightarrow 2)$$
$$\mathbf{B}_3(t_{l\mu})=\frac{1}{A_1A_2}\left[\frac{\partial (A_2t_{31})}{\partial\alpha_1}+\frac{\partial (A_1t_{32})}{\partial\alpha_2}\right],\quad \mathbf{K}_3(t_{l\mu})=-K_2t_{11}-K_1t_{22}$$

Taking into account that

$$\frac{\partial}{\partial\gamma}=\frac{1}{\varepsilon}\frac{\partial}{\partial z},\quad \frac{d}{d\alpha_\nu}=\frac{\partial}{\partial\alpha_\nu}+\frac{1}{\lambda\varepsilon}\frac{\partial}{\partial y_\nu}$$

we obtain from (1.1), (2.1), (2.2)

$$e_{ij}=e_{ij}^{(0)}+\varepsilon e_{ij}^{(1)}+\varepsilon^2e_{ij}^{(2)}+\ldots \tag{2.6}$$

It follows from (1.3) and (2.6)

$$\sigma_{ij}=\sigma_{ij}^{(0)}+\varepsilon\sigma_{ij}^{(1)}+\varepsilon^2\sigma_{ij}^{(2)}+\ldots \tag{2.7}$$
$$\sigma_{ij}^{(m)}=a_{ijln}(y,z)\,e_{ln}^{(m)}\ (m=0,1,2,\ldots)$$

Taking (2.2) and (2.4) into account, (1.2) is expanded in powers of ε^m $(m=-1,0,1,2)$ in the following manner

$$\frac{1}{\lambda}\frac{\partial\sigma_{i\nu}^{(0)}}{\partial\xi_\nu}+\frac{\partial\sigma_{i3}^{(0)}}{\partial z}=0 \tag{2.8}$$

$$\frac{1}{\lambda}\frac{\partial\sigma_{i\nu}^{(1)}}{\partial\xi_\nu}+\frac{\partial\sigma_{i3}^{(1)}}{\partial z}+\mathbf{B}_i(\sigma_{l\mu}^{(0)})=0$$

$$\frac{1}{\lambda}\frac{\partial}{\partial\xi_\nu}(\sigma_{i\nu}^{(2)}+zK_\nu\sigma_{i\nu}^{(0)})+\frac{\partial}{\partial z}(\sigma_{i3}^{(2)}+zK_3\sigma_{i3}^{(0)})+$$
$$\mathbf{B}_i(\sigma_{l\mu}^{(1)})+\mathbf{K}_i(\sigma_{l\mu}^{(0)})+f_i=0$$

$$\frac{1}{\lambda}\frac{\partial}{\partial\xi_\nu}(\sigma_{i\nu}^{(3)}+zK_\nu\sigma_{i\nu}^{(1)})+\frac{\partial}{\partial z}(\sigma_{i3}^{(3)}+zK_3\sigma_{i3}^{(1)})+$$
$$\mathbf{B}_i(\sigma_{l\mu}^{(2)}+zK_l\sigma_{l\mu}^{(0)})+\mathbf{K}_i(\sigma_{l\mu}^{(1)})+g_i=0$$

Taking account of (2.2) and (2.3), conditions (1.4) are expanded in powers of ε^m ($m = 0, 1, 2, 3$) according to the formulas

$$\sigma_{ij}^{(m)}N_j^\pm=0 \quad (z=z^\pm;\ m=0,1) \tag{2.9}$$

$$(\sigma_{ij}^{(2)}+zK_j\sigma_{ij}^{(0)})N_j^\pm=\pm\omega^\pm r_i^\pm \quad (z=z^\pm) \tag{2.10}$$

$$(\sigma_{ij}^{(3)}+zK_j\sigma_{ij}^{(1)})N_j^\pm=\pm\omega^\pm q_i^\pm \quad (z=z^\pm) \tag{2.11}$$

Here

$$N_j^\pm=\left\{\mp\frac{\partial F^\pm}{\partial\xi_1},\ \mp\frac{\partial F^\pm}{\partial\xi_2},\ 1\right\},\quad \omega^\pm=\left[1+\left(\frac{\partial F^\pm}{\partial\xi_1}\right)^2+\left(\frac{\partial F^\pm}{\partial\xi_2}\right)^2\right]^{1/2} \tag{2.12}$$

In conformity with (2.3) and (2.4), we have

$$f_3=0,\quad g_\nu=0,\quad r_3^\pm=0,\quad q_\nu^\pm=0,\quad \frac{\partial}{\partial\xi_\nu}=\frac{1}{A_\nu}\frac{\partial}{\partial y_\nu},\quad \xi=(\xi_1,\xi_2)$$

We note that in deriving relations (2.6)-(2.12) we took into account that the Lamé coefficients and the components of the unit normals in the coordinate system introduced are expanded in ε.

3. We obtain from relationships (1.1), (2.1), (2.7)

$$\sigma_{ij}^{(0)}=a_{ijl\nu}\frac{1}{\lambda}\frac{\partial u_l^{(1)}}{\partial\xi_\nu}+a_{ijl3}\frac{\partial u_l^{(1)}}{\partial z}+a_{ijl\nu}\omega_{l\nu}^{(0)} \tag{3.1}$$

$$\omega_{11}^{(0)}=\frac{1}{A_1}\frac{\partial u_1^{(0)}}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u_2^{(0)},\quad \omega_{31}^{(0)}=\frac{1}{A_1}\frac{\partial u_3^{(0)}}{\partial\alpha_1}\quad (1\leftrightarrow 2) \tag{3.2}$$

$$\omega_{12}^{(0)}=\omega_{21}^{(0)}=\frac{1}{2}\left[\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{u_1^{(0)}}{A_1}\right)+\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{u_2^{(0)}}{A_2}\right)\right]$$

We use the notation

$$\mathbf{L}_{ijl}=a_{ijl\nu}\frac{1}{\lambda}\frac{\partial}{\partial\xi_\nu}+a_{ijl3}\frac{\partial}{\partial z},\quad \mathbf{D}_{il}=\frac{1}{\lambda}\frac{\partial}{\partial\xi_\beta}\mathbf{L}_{i\beta l}+\frac{\partial}{\partial z}\mathbf{L}_{i3l} \tag{3.3}$$

$$A_{in\nu}=\frac{1}{\lambda}\frac{\partial a_{i\beta n\nu}}{\partial\xi_\beta}+\frac{\partial a_{i3n\nu}}{\partial z} \tag{3.4}$$

Substituting (3.1) into the first equations in (2.8), we obtain

$$\mathbf{D}_{il}u_l^{(1)}=-A_{in\nu}\omega_{n\nu}^{(0)} \tag{3.5}$$

The solutions of (3.5) periodic in ξ_1, ξ_2 should satisfy conditions (2.9) for $m=0$, which can be written thus by using (3.1)-(3.3):

$$(\mathbf{L}_{ijl}u_l^{(1)}+a_{ijn\nu}\omega_{n\nu}^{(0)})N_j^\pm=0 \quad (z=z^\pm) \tag{3.6}$$

We represent the solution of (3.5), (3.6) in the form

$$u_l^{(1)}=U_l^{n\nu}(\xi,z)\,\omega_{n\nu}^{(0)}(\alpha)+v_l(\alpha) \tag{3.7}$$

where $U_l^{n\nu}(\xi,z)$ are solutions periodic in ξ_1, ξ_2 (with periods A_1, A_2, respectively) for the problem

$$\mathbf{D}_{il}U_l^{n\nu}=-A_{in\nu} \tag{3.8}$$
$$b_{ij}^{n\nu}N_j^\pm=0\quad (z=z^\pm);\quad b_{ij}^{n\nu}=\mathbf{L}_{ijl}U_l^{n\nu}+a_{ijn\nu}$$

For $n,\nu = 3,1$ and 3.2 this problem is solved exactly

$$U_1^{31} = -z,\ U_2^{31} = U_3^{31} = 0,\quad U_2^{32} = -z,\ U_1^{32} = U_3^{32} = 0 \tag{3.9}$$

It follows from (3.9)

$$b_{ij}^{3\nu} = 0 \tag{3.10}$$

Substituting (3.7) into (3.1) and taking account of (3.10), we obtain

$$\sigma_{ij}^{(0)} = b_{ij}^{\mu\nu}\omega_{\mu\nu}^{(0)} \tag{3.11}$$

By using integration over the periodicity cell Ω, we introduce the operation of taking the average in the variables y_1, y_2, z

$$\langle\varphi(\alpha, y, z)\rangle = \frac{1}{V}\int_\Omega \varphi(\alpha, y, z)\, dy_1\, dy_2\, dz \tag{3.12}$$

We note that differentiation with respect to the variables α_1 and α_2 is commutative with the averaging operations (3.12), and V is the volume of the cell Ω.

We take the average of the second equation in (2.8) by using conditions (2.9) for $m = 1$ and the periodicity in y_1 and y_2

$$\mathbf{B}_i\,\langle\sigma_{i\mu}^{(0)}\rangle = 0 \tag{3.13}$$

Taking (3.11) into account for $m = 0$ the homogeneous Eqs.(3.13) and conditions (2.9) have the zero solution $\omega_{11}^{(0)} = \omega_{22}^{(0)} = \omega_{12}^{(0)} = 0$. We obtain from (3.2), (3.7) and (3.9)

$$u_1^{(0)} = u_2^{(0)} = 0,\quad u_3^{(0)} = w(\alpha) \tag{3.14}$$

$$u_\nu^{(1)} = v_\nu(\alpha) - z\frac{1}{A_\nu}\frac{\partial w}{\partial\alpha_\nu},\quad u_3^{(1)} = v_3(\alpha)$$

The function $w(\alpha)$ introduced in (3.14) and also the function $v_i(\alpha)$ of problem (3.5), (3.6) are not determined. It hence follows from (3.11) that

$$\sigma_{ij}^{(0)} = 0 \tag{3.15}$$

4. Taking account of (3.14), we obtain from (1.1), (2.1) and (2.7)

$$\sigma_{ij}^{(1)} = \mathbf{L}_{ijl}u_l^{(2)} + a_{ijl\nu}\omega_{l\nu} + z a_{ij\mu\nu}\tau_{\mu\nu} \tag{4.1}$$

$$\omega_{11} = \frac{1}{A_1}\frac{\partial v_1}{\partial\alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}v_2 + K_2 w,\quad \omega_{31} = \frac{1}{A_1}\frac{\partial v_3}{\partial\alpha_1}\quad (1\leftrightarrow 2) \tag{4.2}$$

$$\omega_{12} = \omega_{21} = \frac{1}{2}\left[\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{v_1}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{v_2}{A_2}\right)\right]$$

$$\tau_{11} = -\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}\right) - \frac{1}{A_1A_2^2}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_2}\quad (1\leftrightarrow 2) \tag{4.3}$$

$$\tau_{12} = \tau_{21} = -\frac{1}{A_1A_2}\left(\frac{\partial^2 w}{\partial\alpha_1\partial\alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial w}{\partial\alpha_2}\right)$$

Substituting (3.15) and (4.1) into the second equation of (2.8) and condition (2.9), for $m = 1$, we obtain

$$\mathbf{D}_{il}u_l^{(2)} = -A_{in\nu}\omega_{n\nu} - (a_{i3\mu\nu} + zA_{i\mu\nu})\tau_{\mu\nu} \tag{4.4}$$

$$(\mathbf{L}_{ijl}u_l^{(2)} + a_{ijn\nu}\omega_{n\nu} + z a_{ij\mu\nu}\tau_{\mu\nu})N_j^\pm = 0\quad (z = z^\pm)$$

We represent the solution of (4.4), periodic in ξ_1, ξ_2 in the form

$$u_l^{(2)} = U_l^{n\nu}\omega_{n\nu}(\alpha) + V_l^{\mu\nu}\tau_{\mu\nu}(\alpha) \tag{4.5}$$

where $U_l^{n\nu}(\xi, z)$ is the solution of the local problem (3.8) and $V_l^{\mu\nu}(\xi, z)$ are periodic solutions in ξ_1, ξ_2 for the problems

$$\mathbf{D}_{il}V_l^{\mu\nu} = -a_{i3\mu\nu} - zA_{i\mu\nu} \tag{4.6}$$

$$c_{ij}^{\mu\nu}N_j^\pm = 0\quad (z = z^\pm);\quad c_{ij}^{\mu\nu} = \mathbf{L}_{ijl}V_l^{\mu\nu} + z a_{ij\mu\nu}$$

We note that the functions $A_1(\alpha)$ and $A_2(\alpha)$ occur (3.8) and (4.6) and therefore the variables α_1 and α_2 also occur as parameters. Using relationship (4.5), we obtain from (4.1)

$$\sigma_{ij}^{(1)} = b_{ij}^{\mu\nu}\omega_{\mu\nu} + c_{ij}^{\mu\nu}\tau_{\mu\nu} \tag{4.7}$$

Following (3.12), we take the average of (3.8) and (4.6) which have first been multiplied by z and z^2. Using the periodicity in y_1 and y_2 we obtain

$$\langle b_{3j}^{\mu\nu}\rangle = \langle z b_{3j}^{\mu\nu}\rangle = \langle c_{3j}^{\mu\nu}\rangle = \langle z c_{3j}^{\mu\nu}\rangle = 0 \tag{4.8}$$

Taking account of (4.8) we obtain from (4.7)

$$\langle\sigma_{3j}^{(1)}\rangle = 0,\quad \langle z\sigma_{3j}^{(1)}\rangle = 0 \tag{4.9}$$

5. We take the average of the last two equations of (2.8). Using the periodicity in y_ν, conditions (2.10) and (2.11), and relations (2.12), (3.15) and (4.9), we obtain

$$\mathbf{B}_\nu \langle \sigma_{\beta\delta}^{(1)} \rangle + r_\nu + \langle f_\nu \rangle = 0 \tag{5.1}$$

$$\mathbf{B}_\nu \langle \sigma_{\beta\delta}^{(2)} \rangle = 0 \tag{5.2}$$

$$\mathbf{B}_3 \langle \sigma_{3\mu}^{(2)} \rangle + \mathbf{K}_3 \langle \sigma_{\beta\delta}^{(1)} \rangle + q_3 + \langle g_3 \rangle = 0 \tag{5.3}$$

$$r_\nu(\alpha) = \frac{1}{V} \int_0^1 \int_0^1 (\omega^+ r_\nu^+ + \omega^- r_\nu^-)\, dy_1\, dy_2, \quad q_3(\alpha) = \frac{1}{V} \int_0^1 \int_0^1 (\omega^+ q_3^+ + \omega^- q_3^-)\, dy_1\, dy_2$$

We take the average of the third equation in (2.8) for $i = 1, 2$ after first multiplying by z

$$\mathbf{B}_\mu \langle z\sigma_{\beta\delta}^{(1)} \rangle - \langle \sigma_{3\mu}^{(2)} \rangle + \rho_\mu + \langle z f_\mu \rangle = 0 \tag{5.4}$$

$$\rho_\mu(\alpha) = \frac{1}{V} \int_0^1 \int_0^1 (z^+ \omega^+ r_\mu^+ + z^- \omega^- r_\mu^-)\, dy_1\, dy_2$$

Using (5.4), we eliminate the terms $\langle \sigma_{3\mu}^{(2)} \rangle$ from (5.3)

$$\frac{1}{A_1 A_2} \frac{\partial}{\partial \alpha_\mu} (\mathbf{B}_\mu \langle z\sigma_{\beta\delta}^{(1)} \rangle + \rho_\mu + \langle z f_\mu \rangle) - K_2 \langle \sigma_{11}^{(1)} \rangle - K_1 \langle \sigma_{22}^{(1)} \rangle + q_3 + \langle g_3 \rangle = 0 \tag{5.5}$$

Only $\langle \sigma_{\beta\delta}^{(1)} \rangle$ and $\langle z\sigma_{\beta\delta}^{(1)} \rangle$, occur in (5.1) and (5.5), for which we obtain, by taking the average of (4.7),

$$\langle \sigma_{\beta\delta}^{(1)} \rangle = \langle b_{\beta\delta}^{\mu\nu} \rangle\, \omega_{\mu\nu} + \langle c_{\beta\delta}^{\mu\nu} \rangle\, \tau_{\mu\nu} \tag{5.6}$$

$$\langle z\sigma_{\beta\delta}^{(1)} \rangle = \langle z b_{\beta\delta}^{\mu\nu} \rangle\, \omega_{\mu\nu} + \langle z c_{\beta\delta}^{\mu\nu} \rangle \tau_{\mu\nu}$$

We note that only three functions $w(\alpha)$, $v_1(\alpha)$, $v_2(\alpha)$ occur in (5.6) in terms of $\omega_{\mu\nu}$, $\tau_{\mu\nu}$ by virtue of (4.2) and (4.3).

Substituting (5.6) into (5.1) and (5.5), we obtain a system of three constitutive equations in the functions $w(\alpha)$, $v_1(\alpha)$, $v_2(\alpha)$, which govern the principal terms of the displacement vector and the stress tensor by means of (3.14) and (4.7).

The coefficients $\langle b_{\beta\delta}^{\mu\nu} \rangle$, $\langle z b_{\beta\delta}^{\mu\nu} \rangle$, $\langle c_{\beta\delta}^{\mu\nu} \rangle$ and $\langle z c_{\beta\delta}^{\mu\nu} \rangle$ are the effective stiffness moduli of the average shell which are determined from the solutions of the local problems (3.8) and (4.6). We note that the functions $A_1(\alpha)$ and $A_2(\alpha)$ occur in these problems in terms of the coordinates ξ_1, ξ_2. Therefore, if these functions are not constants, the effective moduli depend on the coordinates α_1, α_2. This means that even in the case of an initially homogeneous material, a "constructive" inhomogeneity can occur as a result of taking the average.

6. We set up a connection between the model constructed and the theory of thin shells. Using the notation from /6, 7/ for the forces, moments, and transverse forces, and taking account of (3.15) and (4.9), we obtain

$$T_\beta = \varepsilon^2 \langle \sigma_{\beta\beta}^{(1)} \rangle + \ldots, S = \varepsilon^2 \langle \sigma_{12}^{(1)} \rangle + \ldots, \tag{6.1}$$

$$M_\beta = \varepsilon^3 \langle z\sigma_{\beta\beta}^{(1)} \rangle + \ldots$$

$$H = \varepsilon^3 \langle z\sigma_{12}^{(1)} \rangle + \ldots, N_\mu = \varepsilon^3 \langle \sigma_{3\mu}^{(2)} \rangle + \ldots$$

To deduce the elasticity relationships, we will solve the local problems (3.8) and (4.6). There is no dependence on y_1, y_2 in the consideration of a "smooth" homogeneous shell $F^\pm \equiv 0$, $a_{ijln} = \text{const}$, and the local problems are solved exactly. For instance, in the isotropic case the non-zero solutions of (3.8) (for $(n = 1,2)$ and (4.6) have the form

$$U_3^{11} = U_3^{22} = -\frac{\nu}{1-\nu} z, \quad V_3^{11} = V_3^{22} = -\frac{\nu}{2(1-\nu)} z^2 \tag{6.2}$$

Using (6.2), we obtain

$$\langle b_{11}^{11} \rangle = \langle b_{22}^{22} \rangle = \frac{E}{1-\nu^2}, \quad \langle b_{11}^{22} \rangle = \langle b_{22}^{11} \rangle = \frac{E\nu}{1-\nu^2} \tag{6.3}$$

$$\langle b_{12}^{12} \rangle = \langle b_{12}^{21} \rangle = \frac{E}{2(1+\nu)}, \quad \langle z b_{\beta\delta}^{\mu\nu} \rangle = 0$$

$$\langle c_{\beta\delta}^{\mu\nu} \rangle = 0, \quad \langle z c_{\beta\delta}^{\mu\nu} \rangle = \frac{1}{12} \langle b_{\beta\delta}^{\mu\nu} \rangle$$

Here E is Young's moduls, and ν is Poisson's ratio.

Substituting (6.3) into (5.6), we find

$$\langle \sigma_{11}^{(1)} \rangle = \frac{E}{1-\nu^2} (\omega_{11} + \nu\omega_{22}), \quad \langle z\sigma_{11}^{(1)} \rangle = \frac{E}{12(1-\nu^2)} (\tau_{11} + \nu\tau_{22}) \quad (1 \leftrightarrow 2) \tag{6.4}$$

$$\langle\sigma_{12}^{(1)}\rangle = \frac{E}{2(1+\nu)}2\omega_{12},\ \langle z\sigma_{12}^{(1)}\rangle = \frac{E}{12(1+\nu)}\tau_{12}$$

Comparing relationships (3.14), (4.2), (4.3) with the corresponding shell theory formulas /7/, we obtain

$$\varepsilon_1 = \varepsilon\,\omega_{11},\quad \varepsilon_2 = \varepsilon\,\omega_{22},\quad \omega = \varepsilon 2\omega_{12}$$
$$\varkappa_1 = \tau_{11},\ \varkappa_2 = \tau_{22},\ \tau = \tau_{12} \tag{6.5}$$

Here ε_1, ε_2, ω are the relative elongations and shear of the middle surface while $\varkappa_1$, $\varkappa_2$, τ characterize the bending and torsion of the middle surface associated with the displacements w deflecting the points of the middle surface along its normal.

Substituting (6.4) and (6.5) into (6.1), we obtain formulas connecting the forces and moments to the strains of the middle surface. They agree with those in shell theory /7/. Substitution of (6.4) into (5.1) and (5.5), taking (6.5), (4.2) and (4.3) into account results in the system of equations used in shell theory within the framework of the Mushtari-Donnell-Vlasov model /7/.

7. We will examine two special cases of the model constructed that are of practical importance.

A cylindrical layer of arbitrary shape. In the α_1, α_2 coordinate system, where α_1 is the distance measured along the generator, and α_2 is measured along the directrix of the cylindrical middle surface, we have /6, 7/

$$A_1 = A_2 = 1,\quad k_1 = 0,\quad k_2 = 1/r\,(\alpha_2) \tag{7.1}$$

In the case of a circular cylinder $r(\alpha_2) = \text{const}$.

We will use the notation $R(\alpha_2) = \varepsilon r(\alpha_2)$; then by virtue of (2.2)

$$K_1 = 1/R\,(\alpha_2),\ K_2 = 0$$

We obtain from (4.2) and (4.3)

$$\omega_{11} = \frac{\partial v_1}{\partial\alpha_1},\quad \omega_{22} = \frac{\partial v_2}{\partial\alpha_2} + \frac{w}{R(\alpha_2)}$$
$$\omega_{12} = \frac{1}{2}\left(\frac{\partial v_1}{\partial\alpha_2} + \frac{\partial v_2}{\partial\alpha_1}\right),\quad \tau_{\mu\nu} = -\frac{\partial^2 w}{\partial\alpha_\mu\partial\alpha_\nu} \tag{7.2}$$

Eqs.(5.1) and (5.5) are written in the form

$$\frac{\partial\langle\sigma_{\nu\delta}^{(1)}\rangle}{\partial\alpha_\delta} + r_\nu + \langle f_\nu\rangle = 0$$
$$\frac{\partial^2\langle z\sigma_{\mu\delta}^{(1)}\rangle}{\partial\alpha_\mu\partial\alpha} - \frac{\langle\sigma_{22}^{(1)}\rangle}{R} + \frac{\partial}{\partial\alpha_\mu}(\rho_\mu + \langle zf_\mu\rangle) + q_3 + \langle g_3\rangle = 0 \tag{7.3}$$

The average stresses and moments are expressed in terms of the middle surface strains (7.2) by using the elasticity relationships (5.6). Since A_1 and A_2 are constants in the case under consideration, the functions $b_{\beta\delta}^{\mu\nu}$ and $c_{\beta\delta}^{\mu\nu}$ can only depend on the coordinates y_1, y_2, z, consequently, all the effective moduli in the elasticity relationships are constants. Their specific values depend on the form of the functions $a_{ijkn}(y_1, y_2, z)$ and $F^{\pm}(y_1, y_2)$ and are determined from the solutions of the local problems (3.8) and (4.6).

A plane layer. Let α_1, α_2, γ be Cartesian coordinates, $A_1 = A_2 = 1$, $k_1 = 0$, and $k_2 = 0$. The main formulas are obtained from (7.2) and (7.3) for $R = \infty$. The elasticity relationships (5.6) can be written in the form

$$\langle\sigma_{\nu\delta}^{(1)}\rangle = \langle b_{\nu\delta}^{\beta\mu}\rangle\frac{\partial v_\beta}{\partial\alpha_\mu} - \langle c_{\nu\delta}^{\beta\mu}\rangle\frac{\partial^2 w}{\partial\alpha_\beta\partial\alpha_\mu} \tag{7.4}$$

$$\langle z\sigma_{\mu\delta}^{(1)}\rangle = \langle zb_{\mu\delta}^{\beta\nu}\rangle\frac{\partial v_\beta}{\partial\alpha_\nu} - \langle zc_{\mu\delta}^{\beta\nu}\rangle\frac{\partial^2 w}{\partial\alpha_\beta\partial\alpha_\nu}$$

Since, as in the cylindrical case, all the effective moduli are constants, we obtain by substituting (7.4) into (7.3) (for $R = \infty$)

$$\langle b_{\nu\delta}^{\beta\mu}\rangle\frac{\partial^2 v_\beta}{\partial\alpha_\delta\partial\alpha_\mu} - \langle c_{\nu\delta}^{\beta\mu}\rangle\frac{\partial^3 w}{\partial\alpha_\delta\partial\alpha_\beta\partial\alpha_\mu} + r_\nu + \langle f_\nu\rangle = 0$$
$$\langle zb_{\mu\delta}^{\beta\nu}\rangle\frac{\partial^3 v_\beta}{\partial\alpha_\mu\partial\alpha_\delta\partial\alpha_\nu} - \langle zc_{\mu\delta}^{\beta\nu}\rangle\frac{\partial^4 w}{\partial\alpha_\mu\partial\alpha_\delta\partial\alpha_\beta\partial\alpha_\nu} + \frac{\partial}{\partial\alpha_\mu}(\rho_\mu + \langle zf_\mu\rangle) + q_3 + \langle g_3\rangle = 0 \tag{7.5}$$

In the limit case of a "smooth" isotropic plate the effective moduli are determined from (6.3) and the well-known equations of plate theory are obtained from (7.5).

We note that despite the fact that the local problems (3.8) and (4.6) are described by differential equations, they can also be solved in the case of practical importance of piece-wise-constant functions $a_{ijln}(y_1, y_2, z)$ that model a fibrous material or composite comprised or a periodic system of grains and a material filling the space between them. In this case the following continuity conditions on the grain surface /1, 2/

$$[U_l^{\mu\nu}] = 0, \ [\lambda^{-1} b_{l\delta}^{\mu\nu} n_\delta + b_{l3}^{\mu\nu} n_3] = 0$$
$$[V_l^{\mu\nu}] = 0, \ [\lambda^{-1} c_{l\delta}^{\mu\nu} n_\delta + c_{l3}^{\mu\nu} n_3] = 0 \tag{7.6}$$

must be appended to the local problems.

Here n_i are components of the vector normal to the contact surface, where we have $A_{i\mu\nu} = 0$ in (3.8) and (4.6).

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Translated by M.D.F.

PMM U.S.S.R.,Vol.51,No.1,pp.56-62, 1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

VARIATIONAL METHODS IN THREE-DIMENSIONAL PROBLEMS OF NON-STATIONARY INTERACTION OF ELASTIC BODIES WITH FRICTION*

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Three-dimensional contact problems are examined for the interaction between a moving elastic body and an elastic foundation under friction conditions. The desired friction force and slip fields depend on the time. A boundary value problem is formulated in the velocities and is reduced to a parabolic variational inequality. Its difference approximation is proposed and will be used to provide a foundation for formulating the problem in increments. A number of methods is proposed for the numerical solution of the problem. The time behaviour of the solution of the non-stationary problem is investigated. The non-stationarity effects in contact problems with friction are considered first under conditions of body displacement relative to the foundation /1/. Three-dimensional problems formulated in increments of the desired functions were studied in /2/. Quasistatic problems in increments and dynamic problems on the contact between a stamp and elastic solid of finite size were investigated /3/. The method of reducing the non-stationary parabolic problems to sequences of variational problems (in application to viscoplastic flow problems) was used in /4, 5/.

1. **Kinematic relationships. Boundary conditions.** We examine the motion of an elastic body on an elastic foundation with a plane surface. We consider the velocities of the

**Prikl.Matem.Mekhan.*,51,1,76-83,1987